Wilson fermion doubling phenomenon on irregular lattice: the similarity and difference with the case of regular lattice

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It is shown that the Wilson fermion doubling phenomenon on irregular lattices (simplicial complexes) does exist. This means that the irregular (not smooth) zero or soft modes exist. The statement is proved on 4 Dimensional lattice by means of the Atiyah-Singer index theorem, then it is extended easily into the cases $D < 4$. But there is a fundamental difference between doubled quanta on regular and irregular lattices: in the latter case the propagator decreases exponentially. This means that the doubled quanta on irregular lattice are "bad" quasiparticles.

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I. INTRODUCTION

The Wilson fermion doubling phenomenon on the regular periodic lattices has been discovered long ago in [1]. The phenomenon and its influence on physics was studied in a number of works (for example see [2]-[4]). It was proved in [5]-[6] that the fermion doubling phenomenon indeed takes place on any periodic lattice with local fermion action transforming to the usual Dirac action in long-wavelength region. But the question about the existence of the Wilson fermion doubling on irregular lattices is open at present. This means that the problem is unsolved in the case of lattice quantum gravity theory [14] (see [7]-[8]).

In this paper I show that the Wilson fermion doubling phenomenon on irregular lattices (simplicial complexes) with $D \leq 4$ does exist. However, there exists a fundamental difference between the propagation of doubling modes on regular and irregular lattices. In the first case the propagator of the irregular modes is the same as the propagator of the regular modes from the spectrum origin, i.e. power-behaved. On the contrary, the propagation of irregular modes on irregular lattice is similar to the Markov process of a random walks. Thus the propagator of irregular modes on irregular lattice decreases very quickly (exponentially): the doubled irregular modes are "bad" quasiparticles.

II. FERMIONS ON IRREGULAR LATTICE

First of all, one must outline shortly the Dirac system on the simplicial complexes. More general problem (the definition of Dirac system in discrete lattice gravity) has been solved in [7]-[8]. Here I simplify the problem assuming that the 4-Dimensional simplicial complex $\mathfrak{K}$ is embedded into 4-Dimensional Euclidean space and the curvature and torsion are equal to zero.

Further all definitions and designations are similar to that in [7, 8]. The four Dirac matrices (4×4) satisfy the well known Clifford algebra

$$\gamma^a\gamma^b + \gamma^b\gamma^a = 2\delta^{ab}, \quad \gamma^5 \equiv \gamma^1\gamma^2\gamma^3\gamma^4\,,$$
$$\mathrm{tr}\ \gamma^5\gamma^a\gamma^b\gamma^c\gamma^d = 4\,\varepsilon^{abcd}, \quad \sigma^{ab} \equiv \frac{1}{4}\left[\gamma^a,\,\gamma^b\right]. \qquad (2.1)$$

The vertices of the complex are denoted as $a_{\mathcal{V}}$, the index $\mathcal{V} = 1, 2, \ldots, \mathfrak{N} \to \infty$ enumerates the vertices. Let the index $\mathcal{W}$ enumerates 4-simplices. It is necessary to use the local enumeration of the vertices $a_{\mathcal{V}}$ attached to a given 4-simplex: the all five vertices of a 4-simplex with index $\mathcal{W}$ are enumerated as $a_{\mathcal{W}i}$, $a_{\mathcal{W}j}$, $a_{\mathcal{W}k}$, $a_{\mathcal{W}l}$, and $a_{\mathcal{W}m}$, $i, j, \ldots = 1, 2, 3, 4, 5$. It must be kept in mind that the same vertex, 1-simplex et cetera can belong to the another adjacent 4-simplexes. The later notations with extra index $\mathcal{W}$ indicate that the corresponding quantities belong to the 4-simplex with index $\mathcal{W}$. The Levi-Civita symbol with in pairs different indexes $\varepsilon_{\mathcal{W}ijklm} = \pm 1$ depending on whether the order of vertices $a_{\mathcal{W}i}a_{\mathcal{W}j}a_{\mathcal{W}k}a_{\mathcal{W}l}a_{\mathcal{W}m}$ defines the positive or negative orientation of this 4-simplex. An element of the (isotopic) gauge group

$$U_{\mathcal{W}ij} = U^{-1}_{\mathcal{W}ji} = \exp\left(ieA_{\mathcal{W}ij}\right), \quad A_{\mathcal{W}ij} \in \mathcal{L}, \qquad (2.2)$$

where $\mathcal{L}$ is the Lie algebra of the gauge group, and an

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elementary vector

$$e^a_{\mathcal{W}ij} \equiv -e^a_{\mathcal{W}ji}, \tag{2.3}$$

are assigned for each oriented 1-simplex $a_{\mathcal{W}i}a_{\mathcal{W}j}$. The Dirac spinors $\psi_{\mathcal{V}}$ and $\psi^\dagger_{\mathcal{V}}$ are assigned to each vertex $a_{\mathcal{V}}$. The Dirac spinors and the gauge field $A_{\mathcal{W}ij}$ belong to the same representation of algebra $\mathcal{L}$.

The Euclidean Hermitean action of the Dirac field associated with the complex $\mathfrak{K}$ has the form

$$\mathfrak{A}_\psi = -\frac{1}{3!\,5!)}\sum_{\mathcal{W}}\sum_{i,j,k,l,m}\varepsilon_{\mathcal{W}ijklm}\varepsilon^{abcd}\left(i\,\psi^\dagger_{\mathcal{W}m}\gamma^a U_{\mathcal{W}mi}\psi_{\mathcal{W}i}\right)e^b_{\mathcal{W}mj}e^c_{\mathcal{W}mk}e^d_{\mathcal{W}ml} \equiv$$
$$\equiv \sum_{\mathcal{V}_1\mathcal{V}_2}\psi^\dagger_{\mathcal{V}_1 s_1}\left[-i\gamma^a_{s_1s_2}\mathcal{D}^a_{\mathcal{V}_1,\mathcal{V}_2}\right]\psi_{\mathcal{V}_2 s_2} \equiv \sum_{\mathcal{V}_1\mathcal{V}_2}\psi^\dagger_{\mathcal{V}_1 s_1}\left[-i\,\not{\mathcal{D}}_{\mathcal{V}_1,\mathcal{V}_2}\right]_{s_1s_2}\psi_{\mathcal{V}_2 s_2}. \tag{2.4}$$

The indices $s_1,\, s_2 = 1,2,3,4$ are the Dirac one. The action (2.4) is invariant under the gauge transformations

$$U_{\mathcal{W}ij} \to S_{\mathcal{W}i}\,U_{A\,ij}\,S^{-1}_{\mathcal{W}j}, \quad S_{\mathcal{W}i} \in SU(2),$$
$$\psi_{\mathcal{W}i} \to S_{\mathcal{W}i}\,\psi_{\mathcal{W}i}, \qquad \psi^\dagger_{\mathcal{W}i} \to \psi^\dagger_{\mathcal{W}i}\,S^{-1}_{\mathcal{W}i}. \tag{2.5}$$

The curvature in (2.4) is equal to zero by definition. The system of equations

$$\left(e^a_{ij} + e^a_{jk} + \ldots + e^a_{li}\right) = 0 \tag{2.6}$$

means that the torsion is also zero. Here the sums in the parentheses are taken on any and all closed paths. Therefore the following interpretation is valid: $e^a_{\mathcal{W}ij} = \left(x^a_{\mathcal{W}j} - x^a_{\mathcal{W}i}\right)$, where $x^a_{\mathcal{W}i}$ are the cartesian coordinates of the vertex $a_{\mathcal{W}i}$.

Let

$$v_{\mathcal{W}} = \frac{1}{(4!)(5!)}\varepsilon_{abcd}\varepsilon_{\mathcal{W}ijklm}e^a_{\mathcal{W}mi}e^b_{\mathcal{W}mj}e^c_{\mathcal{W}mk}e^d_{\mathcal{W}ml} \tag{2.7}$$

be the oriented volume of the $\mathcal{W}$-4-simplex and $v_{\mathcal{V}}$ be the sum of the volumes $v_{\mathcal{W}}$ for that $\mathcal{W}$-4-simplexes which contain the vertex $a_{\mathcal{V}}$. Thus the spinor space scalar product is given by

$$\langle\psi_1|\psi_2\rangle = \frac{1}{5}\sum_{\mathcal{V}} v_{\mathcal{V}}\psi^\dagger_{(1)\mathcal{V}}\psi_{(2)\mathcal{V}}. \tag{2.8}$$

The operator $[i\,\not{\mathcal{D}}_{\mathcal{V}_1,\mathcal{V}_2}]$ in (2.4), as well as the operator $\left[i\,(v_{\mathcal{V}_1})^{-1/2}\,\not{\mathcal{D}}_{\mathcal{V}_1,\mathcal{V}_2}\,(v_{\mathcal{V}_2})^{-1/2}\right]$, are Hermitian. Thus the eigenfunction problem

$$\sum_{\mathcal{V}_2}\left[i\left(\frac{1}{\sqrt{v_{\mathcal{V}_1}}}\right)\not{\mathcal{D}}_{\mathcal{V}_1,\mathcal{V}_2}\left(\frac{1}{\sqrt{v_{\mathcal{V}_2}}}\right)\right]\left(\sqrt{v_{\mathcal{V}_2}}\psi_{(\mathfrak{P})\mathcal{V}_2}\right) =$$
$$= \frac{1}{5}\epsilon_{\mathfrak{P}}\left(\sqrt{v_{\mathcal{V}_1}}\psi_{(\mathfrak{P})\mathcal{V}_1}\right) \longleftrightarrow$$
$$\longleftrightarrow \sum_{\mathcal{V}_2}\left[-\frac{i}{v_{\mathcal{V}_1}}\,\not{\mathcal{D}}_{\mathcal{V}_1,\mathcal{V}_2}\right]\psi_{(\mathfrak{P})\mathcal{V}_2} = \frac{1}{5}\epsilon_{\mathfrak{P}}\psi_{(\mathfrak{P})\mathcal{V}_1} \tag{2.9}$$

is correct, and the set of eigenfunctions $\left\{\psi_{(\mathfrak{P})}\right\}$ forms a complete orthonormal basis in the metric (2.8). Let's expand the Dirac fields in this basis:

$$\psi_{\mathcal{V}} = \sum_{\mathfrak{P}}\eta_{\mathfrak{P}}\psi_{(\mathfrak{P})\mathcal{V}}, \quad \psi^\dagger_{\mathcal{V}} = \sum_{\mathfrak{P}}\eta^\dagger_{\mathfrak{P}}\psi^\dagger_{(\mathfrak{P})\mathcal{V}}. \tag{2.10}$$

The new dynamic variables $\{\eta_{\mathfrak{P}},\, \eta^\dagger_{\mathfrak{P}}\}$ are Grassmann. The scalar product (2.8) in these variables is rewritten as

$$\langle\psi_1|\psi_2\rangle = \sum_{\mathfrak{P}}\eta^\dagger_{(1)\mathfrak{P}}\eta_{(2)\mathfrak{P}}. \tag{2.11}$$

It is important here that

$$\gamma^5 i\,\not{\mathcal{D}}_{\mathcal{V}_1,\mathcal{V}_2} = -i\,\not{\mathcal{D}}_{\mathcal{V}_1,\mathcal{V}_2}\gamma^5. \tag{2.12}$$

The long-wavelength limit of the theory is straightforward. To do this one should believe the quantities $A_{\mathcal{W}ij}$ and $e^a_{\mathcal{W}ij}$ as the smooth 1-forms

$$A_{\mathcal{W}ij} \to A_a(x)\mathrm{d}x^a, \quad e^a_{\mathcal{W}ij} \to \mathrm{d}x^a$$

taking the small values $A_{\mathcal{W}ij}$ and $e^a_{\mathcal{W}ij}$ on the vector $e^a_{\mathcal{W}ij}$, and substitute the smooth Dirac field $\psi(x)$ taking the value $\psi_{\mathcal{V}}$ on the vertex $a_{\mathcal{V}}$ for the set of spinors $\psi_{\mathcal{V}}$. As a result the action (2.4), the scalar product (2.8) and the eigenvalue problem (2.9) transform to the well known expressions and equation:

$$\mathfrak{A}_\psi = \int\left(-i\psi^\dagger\gamma^a\nabla_a\psi\right)\mathrm{d}x^1\wedge\mathrm{d}x^2\wedge\mathrm{d}x^3\wedge\mathrm{d}x^4,$$
$$\nabla_a = \partial_a + ieA_a, \tag{2.13}$$

$$\langle\psi_1|\psi_2\rangle = \int\psi^\dagger_1(x)\psi_2(x)\mathrm{d}^{(4)}x, \tag{2.14}$$

$$-i\gamma^a\nabla_a\psi_{(\mathfrak{P})}(x) = \epsilon_{\mathfrak{P}}\psi_{(\mathfrak{P})}(x). \tag{2.15}$$

III. THE GAUGE ANOMALY AND ATIYAH-SINGER INDEX THEOREM

The partition function of the fermion system as the functional of the quantities $\{e^a_{\mathcal{W}ij}\}$ and $\{A_{\mathcal{W}ij}\}$ is given by integral

$$Z\{e^a_{\mathcal{W}ij},\, A_{\mathcal{W}ij}\} = \int \left(D\psi^\dagger D\psi\right) \exp \mathfrak{A}_\psi \,. \qquad (3.1)$$

Here the fermion functional measure is defined according to

$$\left(D\psi^\dagger D\psi\right) \equiv \prod_{\mathcal{V}} \mathrm{d}\psi^\dagger_{\mathcal{V}} \mathrm{d}\psi_{\mathcal{V}} F\{e^a_{\mathcal{W}ij}\}, \qquad (3.2)$$

where

$$d\psi_{\mathcal{V}} = \prod_{\varkappa} \prod_{s=1}^{4} \mathrm{d}\psi_{\mathcal{V}\varkappa s}, \quad d\psi^\dagger_{\mathcal{V}} = \prod_{\varkappa} \prod_{s=1}^{4} \mathrm{d}\psi^\dagger_{\mathcal{V}\varkappa s}, \qquad (3.3)$$

and the index $\varkappa$ enumerates the components of the gauge representation. The functional $F\{e^a_{\mathcal{W}ij}\}$ in (3.2) can be calculated easily with the help of the metric (2.8), but it is not interesting here. The scalar product (2.11) in Grassmann variables $\{\eta_{\mathfrak{P}},\, \eta^\dagger_{\mathfrak{P}}\}$ permits to rewrite the measure (3.2) as below:

$$\left(D\psi^\dagger D\psi\right) = \prod_{\mathfrak{P}} \mathrm{d}\eta^\dagger_{\mathfrak{P}} \mathrm{d}\eta_{\mathfrak{P}}. \qquad (3.4)$$

Let's study the chiral transformation of the Dirac field

$$\psi_{\mathcal{V}} \to \exp\left(i\alpha_{\mathcal{V}}\gamma^5\right)\psi_{\mathcal{V}}, \quad \psi^\dagger_{\mathcal{V}} \to \psi^\dagger_{\mathcal{V}} \exp\left(i\alpha_{\mathcal{V}}\gamma^5\right). \quad (3.5)$$

Obviously, the measure (3.2) is invariant under the transformation (3.5). Moreover, even the factors $\left(\prod_{s=1}^4 d\psi_{\mathcal{V}\varkappa s}\right)$ and $\left(\prod_{s=1}^4 \mathrm{d}\psi^\dagger_{\mathcal{V}\varkappa s}\right)$ of the measure (3.2) each are invariant since the matrix γ^5 is traceless. It follows from here that the measure in right-hand side of Eq. (3.4) is also invariant under the chiral transformation and the corresponding Jacobian $J = 1$. The last statement permits to extract some interesting information.

Suppose the chiral transformation is infinitesimal: $\alpha_{\mathcal{V}} \to 0$. From the linearized transformations of the Dirac field (3.5) we obtain linearized transformations for the variables $\{\eta_{\mathfrak{P}},\, \eta^\dagger_{\mathfrak{P}}\}$:

$$\begin{aligned} \eta_{\mathfrak{P}} &\to \eta_{\mathfrak{P}} + \frac{i}{5}\sum_{\mathfrak{Q}} \eta_{\mathfrak{Q}} \sum_{\mathcal{V}} \alpha_{\mathcal{V}} \upsilon_{\mathcal{V}} \psi^\dagger_{\mathfrak{P}\mathcal{V}}\gamma^5\psi_{\mathfrak{Q}\mathcal{V}}, \\ \eta^\dagger_{\mathfrak{P}} &\to \eta^\dagger_{\mathfrak{P}} + \frac{i}{5}\sum_{\mathfrak{Q}} \eta^\dagger_{\mathfrak{Q}} \sum_{\mathcal{V}} \alpha_{\mathcal{V}} \upsilon_{\mathcal{V}} \psi^\dagger_{\mathfrak{Q}\mathcal{V}}\gamma^5\psi_{\mathfrak{P}\mathcal{V}}. \end{aligned} \qquad (3.6)$$

The Jacobian of this transformation is equal to

$$J = \left(1 + \frac{2i}{5}\sum_{\mathcal{V}} \alpha_{\mathcal{V}} \upsilon_{\mathcal{V}} \sum_{\mathfrak{P}} \psi^\dagger_{\mathfrak{P}\mathcal{V}}\gamma^5\psi_{\mathfrak{P}\mathcal{V}}\right).$$

On the other hand, as was stated before, $J = 1$. Therefore, since the quantities $\alpha_{\mathcal{V}}$ are arbitrary at each vertex, we have

$$\sum_{\mathfrak{P}} \psi^\dagger_{\mathfrak{P}\mathcal{V}}\gamma^5\psi_{\mathfrak{P}\mathcal{V}} = 0. \qquad (3.7)$$

For the following analysis it is necessary to decompose the sum (3.7) into infrared or long-wavelength and the rest ultraviolet parts. Firstly let's consider the infrared part. One must introduce the following scales: the gauge field wavelength order $\sim \lambda$; the scale of ultraviolet cutoff of the long-wavelength sector Λ; the lattice scale $l_P \sim \left|e^a_{\mathcal{W}ij}\right|$. The scales satisfy inequalities

$$\lambda^{-1} \ll \Lambda \ll l_P^{-1}. \qquad (3.8)$$

Let us divide the total index set $\{\mathfrak{P}\}$ into three subsets. For the long-wavelength $\psi_{\mathfrak{P}}(x)$:

$$\begin{aligned} \mathfrak{P} \in \mathcal{S}_{\text{infra}} &\iff |\epsilon_{\mathfrak{P}}| < \Lambda_1, \quad \lambda^{-1} \ll \Lambda_1 \ll l_P^{-1}, \\ \mathfrak{P} \in \mathcal{S}^{\circledcirc}_{\text{infra}} &\iff \Lambda_1 < |\epsilon_{\mathfrak{P}}| < \Lambda_2 \ll l_P^{-1}. \end{aligned}$$

The rest of indexes is designated as $\mathcal{I}$, so that

$$\mathcal{S}_{\text{infra}} + \mathcal{S}^{\circledcirc}_{\text{infra}} + \mathcal{I} = \{\mathfrak{P}\}$$

In consequence of Eq. (2.12) it is evident that for all $\mathfrak{P}$ with $\epsilon_{\mathfrak{P}} \neq 0$ (see Eq. (2.9))

$$\frac{1}{5}\sum_{\mathcal{V}} \upsilon_{\mathcal{V}} \psi^\dagger_{\mathfrak{P}\mathcal{V}}\gamma^5\psi_{\mathfrak{P}\mathcal{V}} = 0. \qquad (3.9)$$

Due to Eq. (3.9) and the identity $\gamma^5 \equiv \left[(1+\gamma^5)/2 - (1-\gamma^5)/2\right]$ we obtain the relation

$$\frac{1}{5}\sum_{\mathcal{V}} \upsilon_{\mathcal{V}} \sum_{\mathfrak{P}\in\mathcal{S}} \psi^\dagger_{\mathfrak{P}\mathcal{V}}\gamma^5\psi_{\mathfrak{P}\mathcal{V}} = n^{\mathcal{S}}_+ - n^{\mathcal{S}}_-, \qquad (3.10)$$

where $\mathcal{S}$ is a subset of the index set $\{\mathfrak{P}\}$ and $n^{\mathcal{S}}_+$ $\left(n^{\mathcal{S}}_-\right)$ is the number of right (left) zero modes on the index subset $\mathcal{S}$. In any case the value of the left-hand side of Eq. (3.10) is a whole number $0, \pm 1, \ldots$.

The value of the long-wavelength part of the sum (3.7) is well known [15]:

$$\begin{aligned} \sum_{\mathfrak{P}\in\mathcal{S}_{\text{infra}}} \psi^\dagger_{\mathfrak{P}}(x)\gamma^5\psi_{\mathfrak{P}}(x) &= -\frac{e^2}{32\pi^2}\varepsilon^{abcd}\,\mathrm{tr}\left\{F_{ab}(x)F_{cd}(x)\right\} + \\ &+\mathrm{O}\left(\frac{1}{(\lambda\Lambda_1)^2}\right)\mathcal{F}_1\{A\} + \mathrm{O}\left(\frac{l_P}{\lambda}\right)\mathcal{F}_2\{A\}, \\ F_{ab} &= \partial_a A_b - \partial_b A_a + ie\left[A_a,\, A_b\right]. \end{aligned} \qquad (3.11)$$

Here $\mathcal{F}_1\{A\}$ and $\mathcal{F}_2\{A\}$ are some local gauge invariant functionals of the gauge field. The rigorous lattice ex-

pression for the left hand side of Eq. (3.11) looks like

$$\sum_{\mathfrak{P}\in\mathcal{S}_{\text{infra}}} \psi^{\dagger}_{\mathfrak{P}\mathcal{V}}\gamma^5\psi_{\mathfrak{P}\mathcal{V}} = \operatorname{tr}\gamma^5 K_{\mathcal{V},\mathcal{V}}(\Lambda_1),$$

$$K_{\mathcal{V}_1,\mathcal{V}_2}(\Lambda_1) \equiv \sum_{\mathfrak{P}} \exp\left[-\frac{(\epsilon_{\mathfrak{P}})^2}{\Lambda_1^2}\right]\psi_{\mathfrak{P}\mathcal{V}_1}\psi^{\dagger}_{\mathfrak{P}\mathcal{V}_2} = = \exp\left[-\frac{(i\,\not{\mathcal{D}})^2}{\Lambda_1^2}\right]_{\mathcal{V}_1,\mathcal{V}_2}. \tag{3.12}$$

The expansion of the lattice operator $K_{\mathcal{V}_1,\mathcal{V}_2}(\Lambda_1)$ in power series in $(\lambda\Lambda_1)^{-2} \ll 1$ and $(l_P/\lambda) \ll 1$ leads to the expression in the right hand side of Eq. (3.11). It is important that this expansion is correct since the operator $K_{\mathcal{V}_1,\mathcal{V}_2}(\Lambda_1)$ is well defined.

The space integral of the right-hand side of Eq. (3.11) is equal to

$$q + \mathrm{O}\left(\frac{1}{(\lambda\Lambda)^2}\right)c_1 + \mathrm{O}\left(\frac{l_P}{\lambda}\right)c_2,$$
$$\frac{c_1}{(\lambda\Lambda)^2} \to 0, \quad \frac{l_P c_2}{\lambda} \to 0. \tag{3.13}$$

Here $q = 0, \pm1, \ldots$ is the topological charge of the gauge field instanton and the numbers c_1 and c_2 tend to some finite values in the limit $(1/\lambda\Lambda) \to 0$ and $(l_P/\lambda) \to 0$. Since the value of the left-hand side of Eq. (3.11) is a whole number (see Eq. (3.10)) and the latter two summands in (3.13) are negligible in comparison with 1, so one must conclude that $c_1 = c_2 = 0$. Finally we have:

$$\frac{1}{5}\sum_{\mathcal{V}} v_{\mathcal{V}} \sum_{\mathfrak{P}\in\mathcal{S}_{\text{infra}}} \psi^{\dagger}_{\mathfrak{P}\mathcal{V}}\gamma^5\psi_{\mathfrak{P}\mathcal{V}} = q. \tag{3.14}$$

This equation is rigorous for $(1/\lambda\Lambda) \lll 1$, $(l_P/\lambda) \lll 1$. Moreover, it follows from the Eq. (3.11), that

$$\sum_{\mathfrak{P}\in\mathcal{S}_{\text{infra}}} \psi^{\dagger}_{\mathfrak{P}}(x)\gamma^5\psi_{\mathfrak{P}}(x) = = -\frac{e^2}{32\pi^2}\varepsilon^{abcd}\operatorname{tr}\left\{F_{ab}(x)F_{cd}(x)\right\}. \tag{3.15}$$

in the limit $(1/\lambda\Lambda) \to 0$ and $(l_P/\lambda) \to 0$. It is well known that the right-hand side of Eq. (3.15) is a one-half of the axial vector anomaly. Here the expression for the anomaly is extracted from the fermion measure (3.4). This method was suggested by Vergeles [9] and Fujikawa [10].

Note that the value of the sum in (3.15) does not depend on the cutoff parameter Λ if it is enclosed in a range of values (3.8). This fact in turn means that

$$\sum_{\mathfrak{P}\in\mathcal{S}^{\circledcirc}_{\text{infra}}} \psi^{\dagger}_{\mathfrak{P}}(x)\gamma^5\psi_{\mathfrak{P}}(x) = 0. \tag{3.16}$$

It is clear from here that the decomposition of the sum in (3.7) into long-wavelength and ultraviolet parts is well defined.

The comparison of Eqs. (3.10), (3.7), (3.14) and (3.16) leads to the following equality:

$$\frac{1}{5}\sum_{\mathcal{V}} v_{\mathcal{V}} \sum_{\mathfrak{P}\in\mathcal{I}} \psi^{\dagger}_{\mathfrak{P}\mathcal{V}}\gamma^5\psi_{\mathfrak{P}\mathcal{V}} = n_+^{\mathcal{I}} - n_-^{\mathcal{I}} = -q. \tag{3.17}$$

Here $n_+^{\mathcal{I}}$ ($n_-^{\mathcal{I}}$) is the number of the right (left) *irregular* zero modes of Eq. (2.9). The difference between the usual and irregular modes is as follows: For the usual modes and adjacent vertices $a_{\mathcal{W}i}$ and $a_{\mathcal{W}j}$ we have

$$\left|\psi_{(\mathfrak{P})\mathcal{W}i} - \psi_{(\mathfrak{P})\mathcal{W}j}\right| \sim l_P\,\epsilon_{\mathfrak{P}}\left|\psi_{(\mathfrak{P})\mathcal{W}j}\right| \to 0. \tag{3.18}$$

By definition, the irregular modes can not satisfy the estimation (3.18), but they satisfy the estimation

$$\left|\psi^{\mathcal{I}}_{(\mathfrak{P})\mathcal{W}i} - \psi^{\mathcal{I}}_{(\mathfrak{P})\mathcal{W}j}\right| \sim \left|\psi^{\mathcal{I}}_{(\mathfrak{P})\mathcal{W}i}\right| \tag{3.19}$$

at least at a part of vertices. Thus, the usual and irregular modes are well separated not only by the energy $\epsilon_{\mathfrak{P}}$ but also by the "momentum".

It is important that the relations (3.17) are rigorous.

IV. WILSON FERMION DOUBLING PHENOMENON

Let $q \in \mathbb{Z}$ and $\not{\mathcal{D}}^{(q)}_{\mathcal{V}_1,\mathcal{V}_2}$ be the Dirac operator defined on an instanton with the topological charge (q). Denote by $\psi^{\mathcal{I}}_{(0\,\xi)\mathcal{V}}$ the irregular zero mode of Eq. (2.9):

$$\sum_{\mathcal{V}_2}\left[-\frac{i}{v_{\mathcal{V}_1}}\,\not{\mathcal{D}}^{(q)}_{\mathcal{V}_1,\mathcal{V}_2}\right]\psi^{\mathcal{I}}_{(0\,\xi)\mathcal{V}_2} = 0. \tag{4.1}$$

The index ξ enumerates the zero modes.

Now let's denote by $\left[-(i/v_{\mathcal{V}_1})\,\not{\mathcal{D}}^{(\text{free})}_{\mathcal{V}_1,\mathcal{V}_2}\right]$ the free lattice Dirac operator. Free Dirac operator is obtained from the general one by the gauge field elimination: $U_{\mathcal{W}mi} = \exp(ieA_{\mathcal{W}mi}) \to 1$

It is easy to obtain the following estimation:

$$\sum_{\mathcal{V}_2}\left[-\frac{i}{v_{\mathcal{V}_1}}\,\not{\mathcal{D}}^{(\text{free})}_{\mathcal{V}_1,\mathcal{V}_2}\right]\psi^{\mathcal{I}}_{(0\,\xi)\mathcal{V}_2} = \mathrm{O}\left(\frac{e}{\rho}\left|\psi^{\mathcal{I}}_{(0\,\xi)\mathcal{V}_1}\right|\right). \tag{4.2}$$

Here ρ is the scale of the instanton field $A^{(\text{inst})}_{\mathcal{W}mi}$. The proof of (4.2) is based on the estimations

$$A^{(\text{inst})}_{\mathcal{W}mi} \sim (l_P/\rho) \ll 1,$$
$$1 \approx \exp\left(ieA^{(\text{inst})}_{\mathcal{W}mi}\right) - ieA^{(\text{inst})}_{\mathcal{W}mi} = U_{\mathcal{W}mi} + \mathrm{O}\left(\frac{e\,l_P}{\rho}\right),$$

and the fact that the lattice Dirac operator is linear in $U_{\mathcal{W}mi}$. Therefore

$$\not{\mathcal{D}}^{(\text{free})}_{\mathcal{V}_1,\mathcal{V}_2} = \not{\mathcal{D}}^{(q)}_{\mathcal{V}_1,\mathcal{V}_2} + \mathrm{O}\left(\frac{e\,l_P^4}{\rho}\right).$$

Since $\upsilon_{\mathcal{V}_1} \sim l_P^4$, the estimation (4.2) follows from Eq. (4.1).

Let's expand the field configuration $\psi^{\mathcal{I}}_{(0\,\xi)\mathcal{V}}$ in a series of the free Dirac operator eigenfunctions

$$\psi^{\mathcal{I}}_{(0\,\xi)\mathcal{V}} = \sum_{\mathfrak{P}} c_{\mathfrak{P}} \psi^{(\mathrm{free})}_{(\mathfrak{P})\mathcal{V}},$$
$$\sum_{\mathcal{V}_2} \left[-\frac{i}{\upsilon_{\mathcal{V}_1}} \mathcal{D}^{(\mathrm{free})}_{\mathcal{V}_1,\mathcal{V}_2} \right] \psi^{(\mathrm{free})}_{(\mathfrak{P})\mathcal{V}_2} = \epsilon_{\mathfrak{P}} \psi^{(\mathrm{free})}_{(\mathfrak{P})\mathcal{V}_1}. \tag{4.3}$$

Here $c_{\mathfrak{P}}$ are some complex numbers.

We are interested in the irregular modes contribution to the expansion (4.3):

$$\psi^{\mathcal{I}}_{(0\,\xi)\mathcal{V}} = \sum_{\mathfrak{P}'} c_{\mathfrak{P}'} \psi^{(\mathrm{free})\mathcal{I}}_{(\mathfrak{P}')\mathcal{V}} + \dots\,, \tag{4.4}$$

where the indices $\mathfrak{P}'$ enumerate the irregular modes. It is evident that at least some numbers $c_{\mathfrak{P}'}$ in (4.4) are nonzero:

$$c_{\mathfrak{P}'} \neq 0. \tag{4.5}$$

Indeed, the irregular field configuration cannot be expanded in a series of the regular smooth modes only.

The estimation (4.2) and expansion (4.4) allow to do the final conclusion: the Wilson fermion doubling phenomenon on irregular 4-Dimensional lattices does exist. Otherwise, the energy gap of the order of $\epsilon^{\mathcal{I}}_{\mathfrak{P}} \sim 1/l_P$ would be expected to take place in the sector of all irregular modes of the free Dirac operator. As was said, in any case the expansion (4.4) contains the irregular modes of the operator. Thus, the additional contributions of the order of $(c_{\mathfrak{P}'}/l_P)$ would be in the right-hand side of the estimation (4.2), the numbers $c_{\mathfrak{P}'} \neq 0$. But the right hand side of the estimation (4.2) does not depend on the lattice parameter l_P. Thus there are the soft or low energy irregular Dirac modes, the index $\mathfrak{P}'$ in the expansion (4.4) enumerates only the soft modes. The soft irregular eigenfunctions of the free Dirac operator are called here as doubled fermion modes.

It is necessary to notice, that the suggested approach is valid also for the regular lattices or partially regular lattices such as periodic in one dimension and irregular in the rest dimensions.

To prove the existence of Wilson fermion doubling phenomenon on irregular 3-Dimensional lattices let us consider the Dirac action on the Cartesian product of a 3-Dimensional simplicial complex $\mathfrak{K}$ and the set of integers $\mathbb{R}$. As before, I assume that the 3-Dimensional simplicial complex is embedded into 3-Dimensional Euclidean space, the vertexes of the complex are denoted as $a_{\mathcal{V}}$, the index $\mathcal{V} = 1, 2, \dots, \mathfrak{N} \to \infty$ enumerates the vertices and the index $\mathcal{W}$ enumerates 3-simplices. Again it is necessary to use the local enumeration of the vertices $a_{\mathcal{V}}$ attached to a given 3-simplex: the all four vertices of a 3-simplex with index $\mathcal{W}$ are enumerated as $a_{\mathcal{W}i}$, $i, j, \ldots = 1, 2, 3, 4$. Later the notations with extra index $\mathcal{W}$ indicate that the corresponding quantities belong to the 3-simplex with index $\mathcal{W}$. The Levi-Civita symbol with in pairs different indexes $\varepsilon_{\mathcal{W}lijk} = \pm 1$ depending on whether the order of vertices $a_{\mathcal{W}l}a_{\mathcal{W}i}a_{\mathcal{W}j}a_{\mathcal{W}k}$ defines the positive or negative orientation of this 3-simplex. For each oriented 1-simplex $a_{\mathcal{W}i}a_{\mathcal{W}j}$ of the simplicial complex an elementary vector

$$e^{\alpha}_{\mathcal{W}ij} \equiv -e^{\alpha}_{\mathcal{W}ji}, \quad \alpha,\, \beta,\, \gamma = 1,\, 2,\, 3$$

is assigned. The vector $e^{\alpha}_{\mathcal{W}ij}$ connects the vertex $a_{\mathcal{W}i}$ with the vertex $a_{\mathcal{W}j}$ in 3D Euclidean space. The rest of notations are evident and they are similar to that in the beginning of Section 2, but they are supplied here by the additional index $n = 0, \pm 1, \ldots \in \mathbb{R}$ since the dynamic variables are defined now on the discrete set $\mathfrak{K} \times \mathbb{R}$.

The Euclidean Hermitean action of the Dirac field associated with the set $\mathfrak{K} \times \mathbb{R}$ has the form

$$\mathfrak{A}_{\psi} = -\frac{1}{2!\,4!)} \sum_n \sum_{\mathcal{W}} \sum_{i,j,k,l} \varepsilon_{\mathcal{W}lijk}\, \varepsilon^{\alpha\beta\gamma} \left(i\, \psi^{\dagger}_{\mathcal{W}l,\,n} \gamma^{\alpha} \psi_{\mathcal{W}i,\,n} \right) e^{\beta}_{\mathcal{W}lj,\,n} e^{\gamma}_{\mathcal{W}lk,\,n} - \frac{1}{2} \sum_n \sum_{\mathcal{V}} \upsilon_{\mathcal{V}} \left(i\, \psi^{\dagger}_{\mathcal{V},\,n} \gamma^4 (\psi_{\mathcal{V},\,n+1} - \psi_{\mathcal{V},\,n-1}) \right) =$$
$$= \sum_n \sum_{\mathcal{V}_1\mathcal{V}_2} \psi^{\dagger}_{\mathcal{V}_1 n} \left[-i\gamma^{\alpha} \mathcal{D}^{\alpha}_{\mathcal{V}_1,\mathcal{V}_2} \right] \psi_{\mathcal{V}_2 n} + \sum_{\mathcal{V}} \upsilon_{\mathcal{V}} \sum_{n,n'} \psi^{\dagger}_{\mathcal{V},\,n} \left[-i\, \gamma^4 D_{n,n'} \right] \psi_{\mathcal{V},\,n'}. \tag{4.6}$$

Here $\upsilon_{\mathcal{V}}$ is the total sum of oriented volumes of the adjacent 3-simplices with common vertex $a_{\mathcal{V}}$. The eigenfunction problem (2.9) for irregular modes now looks like

$$\sum_{\mathcal{V}_2} \left[-\frac{i}{\upsilon_{\mathcal{V}_1}} \gamma^{\alpha} \mathcal{D}^{\alpha}_{\mathcal{V}_1,\mathcal{V}_2} \right] \psi^{\mathcal{I}}_{(\mathfrak{P})\mathcal{V}_2 n} + \sum_{n'} \left[-i\, \gamma^4 D_{n,n'} \right] \psi^{\mathcal{I}}_{(\mathfrak{P})\mathcal{V}_1 n'} = \epsilon_{\mathfrak{P}} \psi^{\mathcal{I}}_{(\mathfrak{P})\mathcal{V}_1 n}, \tag{4.7}$$

or briefly

$$\left\{\gamma^\alpha(-i/v_{\mathcal{V}})\mathcal{D}^\alpha + \gamma^4(-iD)\right\}\psi^{\mathcal{I}}_{(\mathfrak{P})} = \epsilon_{\mathfrak{P}}\psi^{\mathcal{I}}_{(\mathfrak{P})}. \qquad (4.8)$$

Both operators $(-i/v_{\mathcal{V}_1})\mathcal{D}^\alpha_{\mathcal{V}_1,\mathcal{V}_2}$ and $-iD_{n,n'}$ are Hermitean and they commute mutually. Therefore, the repeated application of the operator $\left\{\gamma^\alpha(-i/v_{\mathcal{V}_1})\mathcal{D}^\alpha + \gamma^4(-iD)\right\}$ to (4.8) leads to the equation

$$\left\{\left[(i/v_{\mathcal{V}})\mathcal{D}^\alpha\right]^2 + \left[iD\right]^2\right\}\psi^{\mathcal{I}}_{(\mathfrak{P})} = \epsilon^2_{\mathfrak{P}}\psi^{\mathcal{I}}_{(\mathfrak{P})}. \qquad (4.9)$$

due to the fact that $\gamma^\alpha\gamma^4 + \gamma^4\gamma^\alpha = 0$, It has been shown that the soft irregular modes of Eqs. (4.8) and (4.9) do exist, i.e. there exist the eigenvalues of the Eq. (4.8) in the subspace of irregular eigenfunctions of the order of $|\epsilon_{\mathfrak{P}}| \ll l_P^{-1}$. Therefore the spectrum of the operator $[(i/v_{\mathcal{V}})\mathcal{D}^\alpha]$ in the subspace of irregular eigenfunctions contains the eigenvalues of the order of $|\epsilon_{\mathfrak{P}}| \ll l_P^{-1}$. This conclusion follows from Eq. (4.9).

Thus, the doubled fermion modes exist also on 3-Dimensional irregular lattices.

The classification of the doubled fermion modes should be a subject of future scientific research.

V. THE PROPAGATION OF THE IRREGULAR QUANTA

At first let us fix the necessary properties of the usual Dirac propagators

$$iS_c(x-y) \equiv \langle 0|T\psi(x)\overline{\psi}(y)|0\rangle \qquad (5.1)$$

in (3+1) continual space-time with Minkowski signature:

1) the translational and Lorentz invariance;

2) for massless theory

$$\gamma^5 iS_c(x-y) + iS_c(x-y)\gamma^5 = 0; \qquad (5.2)$$

3) for $x^0 > z^0 > y^0$

$$\int \mathrm{d}^{(3)}z \left[iS_c(x-z)\right]\gamma^0\left[iS_c(z-y)\right] = iS_c(x-y); \quad (5.3)$$

4) the propagating particles are "good" quasiparticles, i.e. they live indefinitely and have well defined four-momentum and their energy is positive.

The property 3) is the quantum-mechanical superposition principle and at the same time the property implies that the propagating particle can not be absorbed or created by vacuum, i.e. the particle is distinguishable against the background of the vacuum.

It is easy to see that all four of the properties define uniquely the particle propagator. Indeed, the most general expression for the propagator in the case $x^0 > y^0$ is

$$iS_c(x-y) = \int\left(\frac{\mathrm{d}^{(3)}k}{(2\pi)^3 2|\mathbf{k}|}\right)\times$$
$$\times\left(\gamma^0|\mathbf{k}| - \gamma^\alpha k^\alpha\right)e^{i\mathbf{k}(\mathbf{x}-\mathbf{y}) - i|\mathbf{k}|(x^0-y^0)}f(k^a). \qquad (5.4)$$

Here the measure, the expression in the parentheses and the exponent are Lorentz-invariant. The property 2) is also fulfilled. Since the propagator (5.4) describes the propagation of the real "good" quasiparticles, so the all its dependence on the space-time coordinates $(x-y)$ is given by the exponent. The function $f(k^a)$ in (5.4) also must be Lorentz-invariant. This means that it can depend only on $k^a k_a = 0$ and thus it is constant: $f = C$. The property 3) gives $C^2 = C$. Therefore $f(k^a) = 1$.

If we insist on the properties 1)-2) only and reject the properties 3)-4), then the propagator describes the propagation of some irregular quanta and it can acquire another forms. For example

$$iS_c^{\mathcal{I}}(x-y) \sim l_P^2 i\gamma^a\left(\partial/\partial x^a\right)\delta^{(4)}(x-y). \qquad (5.5)$$

It is shown below that the propagators of the irregular quanta are similar to the expression (5.5). In order to do this, the structure of the fermion vacuum must be described in general.

Now I return to the Euclidean metric. For simplicity, the gauge group is assumed to be trivial, so that the index $\varkappa$ will be omitted. Note that from the integration rules

$$\int \mathrm{d}\psi_{\mathcal{V}s} = 0, \quad \int \mathrm{d}\psi_{\mathcal{V}s}\cdot\psi_{\mathcal{V}s'} = \delta_{s\,s'},$$
$$\int \mathrm{d}\psi^\dagger_{\mathcal{V}s} = 0, \quad \int \mathrm{d}\psi^\dagger_{\mathcal{V}s}\cdot\psi^\dagger_{\mathcal{V}s'} = \delta_{s\,s'} \qquad (5.6)$$

it follows that the nonzero value of the integral (3.1) is obtained only if the complete products of the fermion variable

$$\left(\prod_{s=1}^{4}\psi_{\mathcal{V}s}\psi^\dagger_{\mathcal{V}s}\right) \qquad (5.7)$$

are present at each vertex $a_{\mathcal{V}}$. These products can arise only due to the exponent expansion under the integral (3.1). As a consequence of the expansion the expression $\left\{\psi^\dagger_{\mathcal{V}_1 s_1}\left[-i\,\not{\mathcal{D}}_{\mathcal{V}_1,\mathcal{V}_2}\right]_{s_1 s_2}\psi_{\mathcal{V}_2 s_2}\right\}$ related to the 1-simplex $a_{\mathcal{V}_1}a_{\mathcal{V}_2}$ can appear (see the Dirac action (2.4)) [16]. Let's assign to the corresponding 1-simplex $a_{\mathcal{V}_1}a_{\mathcal{V}_2}$ an arrow in this case. The arrow is vectored from vertex $a_{\mathcal{V}_2}$ to vertex $a_{\mathcal{V}_1}$ which can be designated as $\overrightarrow{a_{\mathcal{V}_2}a_{\mathcal{V}_1}}$ or $\overleftarrow{a_{\mathcal{V}_1}a_{\mathcal{V}_2}}$. Four arrows come into each vertex and four arrows come out from each vertex as a result of integration in (3.1). This geometrical picture is realized analytically by assigning to each 1-simplex $\overleftarrow{a_{\mathcal{V}}a_{\mathcal{V}_1}}$ the matrix $\left[-i\,\not{\mathcal{D}}_{\mathcal{V},\mathcal{V}_1}\right]_{s\,s_1}$ and to each 1-simplex $\overrightarrow{a_{\mathcal{V}}a_{\mathcal{V}_1}}$ the matrix $\left[-i\,\not{\mathcal{D}}_{\mathcal{V}_1,\mathcal{V}}\right]_{s_1 s}$. Thus there is the factor

$$\left\{\sum_{s_1,s_2,s_3,s_4=1}^{4} \varepsilon_{s_1 s_2 s_3 s_4} \left[-i\,\not{\mathcal{D}}_{\mathcal{V},\mathcal{V}_1}\right]_{s_1 s_1'} \left[-i\,\not{\mathcal{D}}_{\mathcal{V},\mathcal{V}_2}\right]_{s_2 s_2'} \left[-i\,\not{\mathcal{D}}_{\mathcal{V},\mathcal{V}_3}\right]_{s_3 s_3'} \left[-i\,\not{\mathcal{D}}_{\mathcal{V},\mathcal{V}_4}\right]_{s_4 s_4'}\right\} \times$$
$$\times \left\{\sum_{s_5,s_6,s_7,s_8=1}^{4} \varepsilon_{s_5 s_6 s_7 s_8} \left[-i\,\not{\mathcal{D}}_{\mathcal{V}_5,\mathcal{V}}\right]_{s_5' s_5} \left[-i\,\not{\mathcal{D}}_{\mathcal{V}_6,\mathcal{V}}\right]_{s_6' s_6} \left[-i\,\not{\mathcal{D}}_{\mathcal{V}_7,\mathcal{V}}\right]_{s_7' s_7} \left[-i\,\not{\mathcal{D}}_{\mathcal{V}_8,\mathcal{V}}\right]_{s_8' s_8}\right\}. \tag{5.8}$$

in every vertex $a_{\mathcal{V}}$

We are interested in the two-point correlator

$$\langle \psi_{\mathcal{V}_1 s_1} \psi^{\dagger}_{\mathcal{V}_2 s_2} \rangle \equiv \frac{\int \left(\mathcal{D}\psi^{\dagger}\mathcal{D}\psi\right) \psi_{\mathcal{V}_1 s_1} \psi^{\dagger}_{\mathcal{V}_2 s_2} \exp \mathfrak{A}_{\psi}}{\int \left(\mathcal{D}\psi^{\dagger}\mathcal{D}\psi\right) \exp \mathfrak{A}_{\psi}}. \tag{5.9}$$

Since there is the external factor $\psi^{\dagger}_{\mathcal{V}_2 s_2}$ in the vertex $a_{\mathcal{V}_2}$, the number of the arrows related with the factors

$$\left[-i\,\not{\mathcal{D}}_{\mathcal{V}_2,\mathcal{V}'}\right]_{s_2 s'} \tag{5.10}$$

and coming into the vertex $a_{\mathcal{V}_2}$ is reduced up to tree. Mathematically this fact is realized by the assigning the inverse matrix

$$\left[-i\,\not{\mathcal{D}}_{\mathcal{V}_2,\mathcal{V}'}\right]^{-1}_{s' s_2},$$
$$\sum_{s'} \left[-i\,\not{\mathcal{D}}_{\mathcal{V}_2,\mathcal{V}'}\right]^{-1}_{s_1 s'} \left[-i\,\not{\mathcal{D}}_{\mathcal{V}_2,\mathcal{V}'}\right]_{s' s_2} = \delta_{s_1,s_2} \tag{5.11}$$

to the corresponding 1-simplex $a_{\mathcal{V}_2} a_{\mathcal{V}'}$ (see Fig. 1). Therefore the number of factors $\psi_{\mathcal{V}' s'}$ presented at the vertex $a_{\mathcal{V}'}$ is reduced up to tree also. To compensate this reduction one must introduce the additional factor (see Fig. 1)

$$\left[-i\,\not{\mathcal{D}}_{\mathcal{V}'',\mathcal{V}'}\right]_{s'' s'}. \tag{5.12}$$

Now the condition at the vertex $a_{\mathcal{V}''}$ is the same as at the beginning of the process at the vertex $a_{\mathcal{V}2}$: the additional factor (5.12) gives an additional arrow coming into the vertex $a_{\mathcal{V}''}$. To eliminate one of them, say $\overleftarrow{a_{\mathcal{V}''} a_{\mathcal{V}'''}}$, one should introduce the factor $\left[-i\,\not{\mathcal{D}}_{\mathcal{V}'',\mathcal{V}'''}\right]^{-1}_{s''' s''}$, and so on. It is evident that the last link in the chain is $\left[-i\,\not{\mathcal{D}}_{\mathcal{V}'''',\mathcal{V}_1}\right]^{-1}_{s_1 s''''}$.

It follows from the above-said that the correlator (5.9) can be represented in the form

$$\langle \psi_{\mathcal{V}_1 s_1} \psi^{\dagger}_{\mathcal{V}_2 s_2} \rangle = \sum_{\text{all paths}} \left\{ \left[-i\,\not{\mathcal{D}}_{\mathcal{V}''''',\mathcal{V}_1}\right]^{-1} \left[-i\,\not{\mathcal{D}}_{\mathcal{V}'''''',\mathcal{V}''''}\right] \ldots \left[-i\,\not{\mathcal{D}}_{\mathcal{V}'',\mathcal{V}'''}\right]^{-1} \left[-i\,\not{\mathcal{D}}_{\mathcal{V}'',\mathcal{V}'}\right] \left[-i\,\not{\mathcal{D}}_{\mathcal{V}_2,\mathcal{V}'}\right]^{-1} \right\}_{s_1 s_2}. \tag{5.13}$$

Obviously, the number of the operators $[-i\mathcal{D}]^{-1}$ is greater than the number of the operators $[-i\,\not{\mathcal{D}}]$ by the unity in the right-hand side of Eq. (5.13). Therefore the total power of the operators $[-i\,\not{\mathcal{D}}]$ and $[-i\,\not{\mathcal{D}}]^{-1}$ in the right-hand side of Eq. (5.13) is odd. Since both these operators are linear in the Dirac matrices γ^a, so the expression in the right-hand side of Eq. (5.13) satisfies the property 2). But the property 3) can not be fulfilled on the microscopic level - if only because of the correlator (5.9) is odd in the total power of the Dirac matrices while the bilinear form of the correlator is even in this sense. Note that a part of information is lost in passing from the microscopic description to the long wavelength limit, and thus the property 3) becomes true. Indeed, the information about the lattice is lost completely in the long wavelength limit and the lattice action (2.4) transforms to the usual continuum Dirac action (2.13). Therefore the correlator (5.9) transforms to the expression (5.4) with $f(k^a) = 1$.

Now let's proceed to the estimation of the irregular quanta correlator. In this case the information related with the lattice is determinative. Because of this, Eq. (5.13) should be used. Since the direct correlator estimation with the help of Eq. (5.13) is impossible, I apply a simple and adequate computational model which describes the problem in terms of continuum theory. Thus the model forgets the details of the lattice.

It is supposed here that the microscopic geometry of the lattice is not fixed. This means that the elementary vectors (2.3) connecting the nearest vertices $a_{\mathcal{W}i}$ and $a_{\mathcal{W}j}$ are quantum variables, so that their quantum fluctuations are described by the corresponding wave function. This point of view is necessary in the lattice quantum theory of gravity [7]-[8]. Though this theory is not satisfactory at present, I hold to the following point of view: if the space-time is discrete on microscopic level, then the corresponding lattice is irregular and the geometrical values describing the lattice are quantum variables. Such

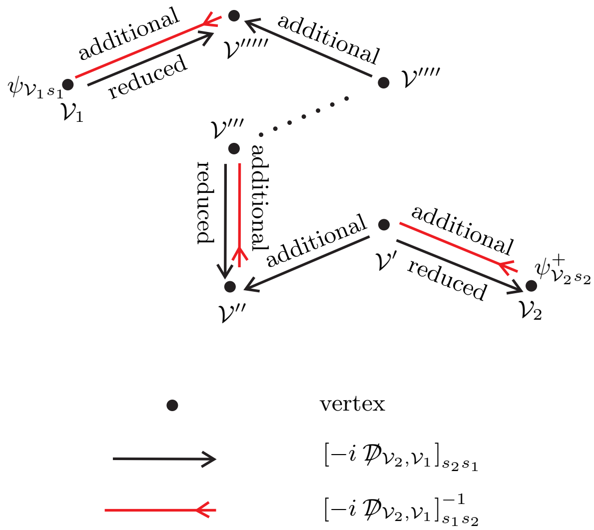

FIG. 1: The graphical representation of the curly brackets in the right-hand side of Eq. (5.13)

lattice is called as "breathing" one.

It seems that the propagation of an irregular fermion on the considered "breathing" lattice is similar in a sence to the dynamics of a Brownian particle: in the process of successive movements of fermion from one vertex to another the information of a previous jump is forgotten due to the irregularity and "breathing" of the lattice. Thus the propagation of irregular fermions can be described by a slightly modified Markov process which must model the correlator (5.13) in the 4-Dimensional Euclidean space.

It is seen from Eqs. (2.4) and (2.7) that

$$\sum_{a=1}^{4} e^{a}_{\mathcal{V}_1,\mathcal{V}_2}\mathcal{D}^{a}_{\mathcal{V}_1,\mathcal{V}_2} \sim v_{\mathcal{V}_1,\mathcal{V}_2}, \tag{5.14}$$

is the sum of oriented volumes of all 4-simplexes with the common 1-simplex $a_{\mathcal{V}_1}a_{\mathcal{V}_2}$. Therefore, the model of the amplitude $\left[-i\gamma^{a}\mathcal{D}^{a}_{\mathcal{V}_1,\mathcal{V}_2}\right]$ in (5.13) will be the following one:

$$[-i\,\not{\mathcal{D}}_{\mathcal{V}_1,\mathcal{V}_2}] \longrightarrow$$
$$[-i\,\not{\mathcal{D}}(x-y)] \equiv \left[\frac{\rho}{\pi b}\,(-i\gamma^{a}\partial_a)\exp\left(-\frac{(x-y)^2}{b^2}\right)\right]. \tag{5.15}$$

The right-hand side of (5.15) is the amplitude of the jump from the point x into the point y. Here the dimensionless Cartesian coordinates $x^{a} \to x^{a}/l_P$ are used. The numerical constant $b \sim 1$ is a parameter of the model, ρ is an unknown normalization constant which is of no importance. It is seen that the direction of the jump vector $(y-x)$ is unconstrained, but the jump step value is constrained by the Gauss distribution. The model of the inverse amplitude $[-i\,\not{\mathcal{D}}_{\mathcal{V}_1,\mathcal{V}_2}]^{-1}$ is as follows:

$$[-i\,\not{\mathcal{D}}_{\mathcal{V}_1,\mathcal{V}_2}]^{-1} \longrightarrow$$
$$[-i\,\not{\mathcal{D}}(x-y)]^{-1} \equiv \left[\frac{1}{\pi\rho\, b}\,(-i\gamma^{a}\partial_a)\exp\left(-\frac{(x-y)^2}{b^2}\right)\right]. \tag{5.16}$$

Now the analog of the relation (5.11) is the equality

$$\int \mathrm{d}^{(4)}y\,[-i\,\not{\mathcal{D}}(x-y)]^{-1}\,[-i\,\not{\mathcal{D}}(y-x)] = 1. \tag{5.17}$$

Thereby the model of the correlator representation (5.13) looks like ($z_0 = y$)

$$\langle\psi(x)\psi^{\dagger}(y)\rangle^{\mathcal{I}} = \sum_{k=0}^{\infty}\prod_{i=1}^{2k+1}\left\{\int \mathrm{d}^{(4)}z_i\right\}\delta^{(4)}(x-z_{2k+1})$$
$$[-i\,\not{\mathcal{D}}(z_{2k+1}-z_{2k})]^{-1}\,[-i\,\not{\mathcal{D}}(z_{2k}-z_{2k-1})]\ldots$$
$$\ldots[-i\,\not{\mathcal{D}}(z_3-z_2)]^{-1}\,[-i\,\not{\mathcal{D}}(z_2-z_1)]\,[-i\,\not{\mathcal{D}}(z_1-y)]^{-1}.$$

Since the operators $[-i\,\not{\mathcal{D}}]$ and $[-i\,\not{\mathcal{D}}]^{-1}$ are coupled one can put $\rho = 1$. This expression is rewritten by passing to the new integration variables $\tilde{z}_i = z_i - z_{i-1}$, $i = 1,\ldots,2k+1$:

$$\langle\psi(x)\psi^{\dagger}(0)\rangle^{\mathcal{I}} = \sum_{k=0}^{\infty}\prod_{i=1}^{2k+1}\int \mathrm{d}^{(4)}z_i\delta^{(4)}\left(x-\sum_{j=1}^{2k+1}z_j\right)[-i\,\not{\mathcal{D}}(z_{2k+1})]^{-1}\,[-i\,\not{\mathcal{D}}(z_{2k})]\ldots[-i\,\not{\mathcal{D}}(z_2)]\,[-i\,\not{\mathcal{D}}(z_1)]^{-1}.$$

With the help of Eqs. (5.15) and (5.16) the right-hand side of the last relation is rewritten once again:

$$\langle\psi(x)\psi^\dagger(0)\rangle^{\mathcal{I}} = \sum_{k=0}^{\infty}\int\ldots\int \mathrm{d}^{(4)}z_1\ldots\mathrm{d}^{(4)}z_{2k+1}\delta^{(4)}\left(\sum_{i=1}^{2k+1} z_i - x\right)\prod_{i=1}^{2k+1}\left[\frac{1}{\pi b}\left(-i\gamma^a\partial_a\right)\exp\left(-\frac{z_i^2}{b^2}\right)\right] =$$

$$= \sum_{k=0}^{\infty}\int\frac{\mathrm{d}^{(4)}q}{(2\pi)^4}e^{-iqx}\prod_{i=1}^{2k+1}\left[\frac{2}{\pi b^3}\int\left(i\gamma^a z_i^a\right)\exp\left(-\frac{z_i^2}{b^2}+iqz_i\right)\mathrm{d}^{(4)}z_i\right] =$$

$$= \int\frac{\mathrm{d}^{(4)}q}{(2\pi)^4}e^{-iqx}\sum_{k=0}^{\infty}\left[2\pi b\left(\gamma^a\frac{\partial}{\partial q^a}\right)\exp\left(-\frac{q^2b^2}{4}\right)\right]^{2k+1} = \left(-i\gamma^a\frac{\partial}{\partial x^a}\right)\int\frac{\mathrm{d}^{(4)}q}{(2\pi)^4}\frac{\pi b^3\exp\left(-\frac{q^2b^2}{4}-iqx\right)}{1-\pi^2b^6q^2\exp\left(-\frac{q^2b^2}{2}\right)}. \quad (5.18)$$

Integral in the right-hand side of Eq. (5.18) is determined for

$$0 < b < \left(\frac{e}{2\pi^2}\right)^{1/4} \approx 0,61. \quad (5.19)$$

Integration over the angle variables leads to the expression ($r \equiv |x|$)

$$\langle\psi(x)\psi^\dagger(0)\rangle^{\mathcal{I}} = \left(-i\gamma^a\frac{\partial}{\partial x^a}\right)$$
$$\left[\left(\frac{b^3}{4\pi r}\right)\int_0^\infty \mathrm{d}q\cdot q^2\frac{J_1(qr)\exp\left(-\frac{q^2b^2}{4}\right)}{1-\pi^2b^6q^2\exp\left(-\frac{q^2b^2}{2}\right)}\right]. \quad (5.20)$$

The characteristic value of the variable q saturating the integral (5.20) is determined by the nearest zero of the denominator in the integral. So $|q| \sim 1$. Since we are interested in the correlator behavior for $r \gg 1$, the argument qr of the Bessel function under the integral (5.20) is effectively large: $qr \gg 1$. Therefore one can use the asymptotic behavior of the Bessel function:

$$J_1(qr) \to \frac{1}{\sqrt{2\pi qr}}\left[e^{iqr-3\pi i/4}+e^{-iqr+3\pi i/4}\right].$$

With the help of the last relation the integral (5.20) is rewritten as follows:

$$\langle\psi(x)\psi^\dagger(0)\rangle^{\mathcal{I}} = \left(-i\gamma^a\frac{\partial}{\partial x^a}\right)\left[\frac{b^3}{2(2\pi r)^{3/2}}\int_C \mathrm{d}q\cdot q^{3/2}\frac{\exp\left(-\frac{q^2b^2}{4}+iqr-3\pi i/4\right)}{1-\pi^2b^6q^2\exp\left(-\frac{q^2b^2}{2}\right)}\right]. \quad (5.21)$$

The integration contour C is pictured on Fig. 2.

We are interested in the denominator zeros in the upper half plane of the complex variable $q = q' + iq''$. The zeros are determined by the following set of equations:

$$(q'^2 - q''^2) = 2q'q''\,\mathrm{ctg}(b^2q'q''),$$
$$2\pi^2b^4\exp[-(b^2q'q'')\,\mathrm{ctg}(b^2q'q'')] = \frac{\sin(b^2q'q'')}{b^2q'q''}. \quad (5.22)$$

Since the solutions of the set of equations (5.22) are symmetrized relative to the imaginary axis, it is enough to solve the system for $q' > 0,\ \ q'' > 0$. The approximative solution of the last set of equations looks like

$$b^2q'q'' \approx (2n+1/2)\,\pi, \quad n = 0, 1, \ldots,$$
$$q'_n \sim q''_n \approx \frac{\sqrt{(2n+1/2)\,\pi}}{b}. \quad (5.23)$$

All zeros of the denominator under the integral (5.21) lead to the simple poles of the expression under the integral sign. Indeed, the derivative of the denominator respect to the integration variable is equal to zero only for $q = 0, \pm\sqrt{2}/b$. Therefore

$$\text{The denominator} = c_n(q - q_n) \ \text{ at } \ q \to q_n.$$

Thus contour C in the integral (5.21) can de deformed up, so that the integral becomes a sum over poles residue. The sum is saturated by the pair of poles which are nearest to the real axis and placed at $q' = \pm\kappa'/b,\ \ q'' = \kappa/b$, where κ', $\kappa \sim 1$ ($n = 0$ in (5.23)).

Finally we have:

$$\langle\psi(x)\psi^\dagger(0)\rangle^{\mathcal{I}} \sim \left(i\gamma^a\frac{\partial}{\partial x^a}\right)\left[\frac{1}{r^{3/2}}\exp(-\kappa r/b)\cos\frac{\kappa' r}{b}\right]. \quad (5.24)$$

The right-hand side of the relation (5.5) simulates the obtained result (5.24) in Minkowski space-time with restored dimensionality.

We see that the irregular quanta are "bad" quasiparticles.

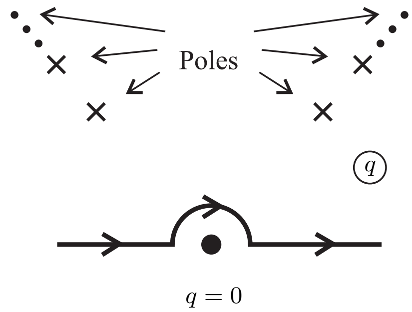

FIG. 2: The integration contour in the integral (5.21) and the location of the integral poles in the complex plane of q-variable

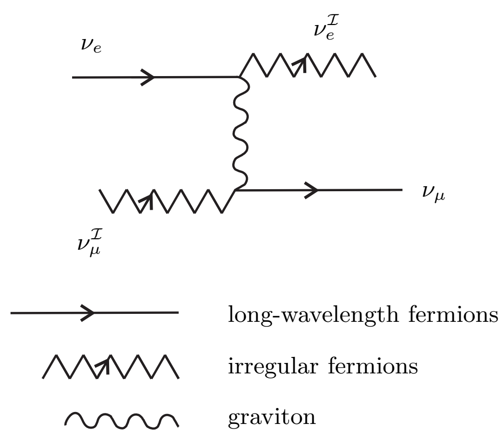

FIG. 3: The process of the electron neutrino transition to the muon one.

The fermion lines, such as in Fig. 1, represent creation (at $\mathcal{V}_2$) , propagation and annihilation (at $\mathcal{V}_1$) of a fermion quantum, and the quantum creation and annihilation events are induced by the external sources only. If the fermion line is everywhere continuous and endless in the space then it describes the propagation of a real particle.

VI. THE MOTIVATION AND SPECULATIONS

Instead of a conclusion I would like to present briefly the motivational factor for the irregular doubled quanta study.

Let us suppose that the space-time is discrete on microscopic level, the corresponding lattice is irregular and "breathing" one (see the previous Section). Suppose also that there are the nonzero densities of the irregular quanta of the three known neutrinos.

The nonzero densities ($n^{\mathcal{I}} \neq 0$) of the irregular quanta does not contradict the fundamental notions of astrophysics since the irregular quanta energy can be arbitrarily small (see the end of Section IV).

The following consequences might have resulted from the suppositions.

I. *The problem of dark matter in cosmology.*

Does the nonzero densities of the neutrino irregular quanta form dark matter in cosmology? It seems that this hypothesis does not contradict to the main properties of dark matter: (i) the irregular quanta are "bad" quasiparticles, so such dark matter can be localized; (ii) the irregular quanta interact very slightly with all normal quanta.

But the nonzero densities of the neutrino irregular quanta give a contribution to the energy-momentum tensor and therefore to the gravitational potential in the vicinity of a metagalaxy.

II. *The problem of the neutrino oscillation.*

The neutrino oscillations, i.e. the mutual oscillating transitions of the neutrinos of different generations, are observed for a long time now. The common explanation of the phenomenon is based on the assumption the neutrino mass matrix is non-diagonal. Moreover, in order to match all the experimental evidences, the extra neutrino fields are introduced, which are sterile regarding to all interactions (naturally, except gravitational one). The sterile neutrinos cannot be observed directly: they are coupled to the three known neutrino generations only by means of a common mass matrix, and this is the way they give a contribution to the neutrino oscillations. The introduction of sterile neutrinos does not exhaust all difficulties of the theory: possibly, the most confounding factor of the theory consists in the fact that the electroweak interaction becomes nonrenormalizable one.

The detailed description of the neutrino oscillations experiments and theory can by found, for example, in [12], [13], and in numerous references there.

Now let's consider the possibility of another physics which may provide the neutrino oscillations. The basis for this physics is the Wilson fermion doubling phenomenon on irregular lattices discussed above.

Let's consider the scattering of the usual normal long-wavelength electron neutrino quantum with the momentum k_e, $|k_e| \ll l_P^{-1}$, by the condensate of muon irregular quanta. Suppose the interaction is mediated by the gravitational field [17]. This scattering process is pictured in Fig. 3. Obviously, time-mean value of the irregular quantum momentum is equal to zero, and the corresponding necessary minimal averaging time $\tau \sim l_P$. This means that the latter has zero momentum in the interaction process of the long-wavelength neutrino quantum with neutrino irregular excitation. Suppose also that vacuum is translation invariant. Then the scattering process in Fig. 3 conserve the momentum of the long-wavelength neutrino: $k_\mu = k_e$. The same process as in Fig. 3 takes place under ν_e and ν_μ interchanging. Finally, we conclude that the neutrino oscillations should be observed since there are mutual transitions of the electron and muon neutrinos with fixed and equal momenta.

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[14] All variants of the lattice gravity theory are defined on the simplicial complexes
[15] Note that the expression in the right hand side of Eq. (3.16) and the integral in the right hand side of Eq. (**??**) are generalized easily into irregular lattice (cimplicial complex) in such a way, that the lattice values transform into the corresponding original continual values in the long-wavelength limit.
[16] By definition of the matrix $[-i\mathcal{D}_{\mathcal{V}_1,\mathcal{V}_2}]_{s_1 s_2}$ the indices $\mathcal{V}_1$ and $\mathcal{V}_2$ enumerate the nearest vertices $a_{\mathcal{V}_1}$ and $a_{\mathcal{V}_2}$, i.e. the vertices belonging to the same 1-simplex $a_{\mathcal{V}_1} a_{\mathcal{V}_2}$.
[17] The interaction mechanism of the scattering process is unclear, but it seems to me that the interaction is mediated by the gravitational field